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ALMOST-FACTORIAL QUARTIC SURFACES WITH A TACNODAL POINT IN $\mathbf{P}^3$

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Abstract

The behaviour of birational transformations between particular surfaces in $\mathbf{P}^3$ and almost-factoriality are investigated. Using a suitable parametrization on $\mathbf{P}^2$ of the quartic surfaces with a tacnodal point, all almost-factorial surfaces of this kind can be classified. We prove that there are nine classes of such surfaces $\mathcal{F}$, and for each of them a possible equation is written and its index of almost-factoriality ν is computed. There are surfaces $\mathcal{F}$ with $\nu = 4, 8, 12$. For each irreducible algebraic curve $\mathcal{C} \subset \mathcal{F}$, we outline how to construct a surface $\mathcal{G}$ such that $\mathcal{F} \cdot \mathcal{G} = \mu\mathcal{C}$, with $\mu \leq \nu$.

1. Introduction

There are classes of algebraic surfaces $\mathcal{F} \subset \mathbf{P}^3$ that have the following property: For every algebraic curve $\mathcal{C} \subset \mathcal{F}$, there is an algebraic surface $\mathcal{S}$ such that $\mathcal{F} \cap \mathcal{S} = \mathcal{C}$, or, more precisely, such that $\mathcal{F} \cdot \mathcal{S} = \mu\mathcal{C}$, where μ is the multiplicity $\mu = I(\mathcal{C}, \mathcal{S} \cap \mathcal{F})$ of intersection

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between $\mathcal{F}$ and $\mathcal{S}$ along $\mathcal{C}$. Such a surface $\mathcal{F}$ is called a *set-theoretic complete intersection surface*; and if $\mathcal{F}$ is non-singular in codimension 1, then it is called *almost-factorial* (*factorial if* $\mu = 1$).

We know that the *index of almost-factoriality* of $\mathcal{F}$ is the integer number ν such that, for every reduced and irreducible curve $\mathcal{C} \subset \mathcal{F}$, there is a surface $\mathcal{S}$ with the property that $\mathcal{S} \cdot \mathcal{F} = \mu\mathcal{C}$, where $\mu \leq \nu$. In this case, we say that $\mathcal{F}$ is ν*-almost-factorial.*

It is well known that the multiplicity of intersection μ between two surfaces $\mathcal{F}$ and $\mathcal{S}$ along a reduced and irreducible curve $\mathcal{C}$ can be computed by considering their affine parts in a suitable affine space $\mathbf{A}^3$ in $\mathbf{P}^3$, such that $\mathcal{C}_a : \mathcal{C} \cap \mathbf{A}^3$ is a curve again. If one of such surfaces $\mathcal{F}$ is normal, the following valuation can be used. Let $\mathcal{F}_a : F = 0$, $\mathcal{G}_a : G = 0$, and $\mathcal{C}_a$ be the affine parts of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{C}$, with $F, G \in k[X, Y, Z]$. Let $k[\mathcal{F}_a] = k[X, Y, Z]/(F) = k[x, y, z]$ and $K = k(\mathcal{F}_a)$ be the quotient field of $k[\mathcal{F}]$, where x, y, z, g denote the canonical projections of the polynomials X, Y, Z, G on $k[\mathcal{F}_a]$. Let $\mathbf{p}$ be the prime ideal of $\mathcal{C}_a$ in $k[\mathcal{F}_a]$, and $\mathbf{v}$ be the valuation centered at $\mathbf{p}$ of the local ring D.V.R. $k[\mathcal{F}_a]_{\mathbf{p}}$. It will defined $\mu = \mathbf{v}(g)$.

Many classes of almost-factorial surfaces in $\mathbf{P}^3$ are well known. It is worth recalling them here.

The planes are factorial. Only the quadrics $\mathcal{F}_2$ with a unique double point (cones) are 2-almost-factorial. Among the cubic surfaces $\mathcal{F}_3$, there are only three families of almost-factorial surfaces $\mathcal{F}_3 \subset \mathbf{P}^3$ (see [13]). Every irreducible quadric cone or cylinder in $\mathbf{A}^3$ is 2-almost-factorial, and every quadric paraboloid in $\mathbf{A}^3$ is factorial. For the affine cubic surfaces $\mathcal{F} \subset \mathbf{A}^3$, there are 82 families of factorial or almost-factorial surfaces (see [4]).

For quartic surfaces $\mathcal{F}_4 \subset \mathbf{P}^3$, we know that the "generic" non-singular surface is factorial (according to Nöther's theorem generalized first by Gröbner; Andreotti and Salmon; and later by Deligne, see [8], [1], [5]), and 33 families of almost-factorial quartic monoids in $\mathbf{P}^3$ have been described (see [11], [12]).

The normal quartic, which is a Zariski's surface $\mathcal{Z}_p : Z^p = G_4(X, Y) \subset \mathbf{A}_k^3$, where k is an algebraically closed field of characteristic $p > 0$ and G_4 is a polynomial of degree ≤ 4, was examined particularly in a book by Lang (see [3]), in which (pages 150-171) the author tackles the factoriality or almost-factoriality of these surfaces.

Biregular birational transformations between algebraic varieties are known to preserve their almost-factoriality (see [2]). A criterion has been given in [6] for the almost-factoriality of V when a birational transformation of $\mathbf{P}^n$ is encountered in a projectively normal variety V.

To the best of our knowledge, nobody knows whether any quartic surfaces in $\mathbf{P}^3$ with only double points on them are factorial or almost-factorial.

The aim of this paper was to exhaustively answer the question of which normal quartic surfaces in $\mathbf{P}^3$ with a tacnodal point on them are almost-factorial.

All the almost-factorial quartic surfaces in $\mathbf{P}^3$ can be placed in 9 classes to within a linear change of coordinates. Using suitable equations for these surfaces $\mathcal{F}$, we adopt a constructive process to obtain a surface $\mathcal{G}$ such that for every curve $\mathcal{C}$ on $\mathcal{F}$, we shall have $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$.

To solve the proposed problem, it is essential to analyze the birational transformations of the surfaces in $\mathbf{P}^3$. In the following paragraphs, k denotes an algebraically closed field of characteristic

$p = 0$, and $\mathbf{A}^3$ and $\mathbf{P}^3$ are the affine and projective spaces on k of dimension 3. A point $(a, b, c) \in \mathbf{A}^3$ will also be identified with $(1 : a : b : c) \in \mathbf{P}^3$.

2. Almost-Factoriality of Surfaces in $\mathbf{P}^3$ and Birational Transformations

Let $F = F(T, X, Y, Z)$ and $G = G(T_1, X_1, Y_1, Z_1)$ be irreducible homogeneous polynomials, and the surfaces $\mathcal{F} : F = 0 \subset \mathbf{P}^3$ and $\mathcal{G} : G = 0 \subset \mathbf{P}_1^3$ be non-singular in codimension 1. Then, let be

$$k[\mathcal{F}] = k[T, X, Y, Z] / J(\mathcal{F}), \quad k[\mathcal{G}] = k[T_1, X_1, Y_1, Z_1] / J(\mathcal{G}).$$

These rings can be regarded as the rings of regular functions of the affine cones over $\mathcal{F}$ and $\mathcal{G}$. They are both integral closed rings because $\mathcal{F}$ and $\mathcal{G}$ are non-singular in codim 1. We must remember that a surface $\mathcal{F} \subset \mathbf{P}^3$ is normal, if it is non-singular in codim 1 and, as a complete intersection of $\mathbf{P}^3$, it is projectively normal to (see [9], Example 84.5, page 188).

Every rational transformation between projective surfaces in $\mathbf{P}^3$ can be regarded as the restriction of suitable transformations of projective space.

Let us consider the following rational transformation: $\tau : \mathbf{P}^3 \longrightarrow \mathbf{P}_1^3$

$$\tau : (T_1 : X_1 : Y_1 : Z_1) = (H_0 : H_1 : H_2 : H_3),$$

given by the four homogeneous polynomials of the same degree

$$H_i \in k[T, X, Y, Z], \quad i = 0, \dots, 3,$$

whose remainders $\bmod J(\mathcal{F})$ does not have a common factor; and the transformation $\sigma : \mathbf{P}_1^3 \longrightarrow \mathbf{P}^3$

$$\sigma : (T : X : Y : Z) = (L_0 : L_1 : L_2 : L_3),$$

given by the four homogeneous polynomials of the same degree

$$L_i \in k[T_1, X_1, Y_1, Z_1], \quad i = 0, \dots, 3,$$

whose remainders $\bmod J(\mathcal{G})$ does not have a common factor. We also have a birational transformation from $\mathcal{F}$ to $\mathcal{G}$, if the following holds:

$$F(H_0, \dots, H_3) \in J(\mathcal{F}), \qquad G(L_0, \dots, L_3) \in J(\mathcal{G}), \tag{1}$$

and there are two non-vanishing polynomials $N \in k[T, X, Y, Z]$ and $N_1 \in k[T_1, X_1, Y_1, Z_1]$, for which the following holds:

$$\sigma \circ \tau : (T_1 : X_1 : Y_1 : Z_1) = (N_1 T_1 : N_1 X_1 : N_1 Y_1 : N_1 Z_1) \bmod J(\mathcal{G}), \tag{2}$$

and

$$\tau \circ \sigma : (T : X : Y : Z) = (NT : NX : NY : NZ) \bmod J(\mathcal{F}). \tag{3}$$

The relations (1), (2), (3) imply that the restrictions of τ on $\mathcal{F} \cap \{N \neq 0\}$ and of σ on $\mathcal{G} \cap \{N_1 \neq 0\}$ are regular maps, one being the inverse of the other.

Indeed, if $P = (t_P : x_P : y_P : z_P) \in \mathcal{F} \cap \{N \neq 0\}$ from (2), then we shall have

$$P = (N(P)t_P : N(P)x_P : N(P)y_P : N(P)z_P) = \sigma(\tau(P)),$$

with $\tau(P) = (L_0(P) : L_1(P) : L_2(P) : L_3(P))$, and with $L_i(P) \neq 0$, for at least one i, $i = 0, \dots, 3$ (otherwise $\sigma((0 : 0 : 0 : 0)) = (0 : 0 : 0 : 0)$ is not the point P). By this and from (1), we obtain $\tau(P) \in \mathcal{G}$.

Below, we call $\mathcal{F} \cap \{N \neq 0\}$ **the set of regularity of** τ.

In the same way, we can see that, if $Q \in \mathcal{G} \cap \{N_1 \neq 0\}$, the point $\sigma(Q)$ exists and $\sigma(Q) \in \mathcal{F}$.

Proposition 1. *Let* $\tau : \mathcal{F} \longrightarrow \mathcal{G}$ *be a birational transformation between two normal algebraic surfaces in* $\mathbf{P}^3$, *and let* $\mathcal{G}$ *be almost-factorial.* $\mathcal{F}$ *is almost factorial if and only if every irreducible curve* $\mathcal{D} \subset \mathcal{F}$, *whose image is a point on* $\mathcal{G}$ *is a set-theoretic complete intersection of* $\mathcal{F}$.

Proof. For τ we keep the previously notations. The assumption that $\dim \mathcal{F} = \dim \mathcal{G} = 2$ implies that τ gives a birational transformation that induces a k isomorphism τ^* between the rational fields on $\mathcal{G}$ and $\mathcal{F}$. We can apply "Zariski's main theorem" (see [10], page 49 in the form given by Bourbaki, Chapter 5, Examples 4-7) to the restriction of the birational transformation τ between the affine varieties $X^2 = \mathcal{F} \cap \{N \neq 0\}$ and $Y^2 = \mathcal{G} \cap \{N_1 \neq 0\}$ of $\mathbf{P}^3$ because X^2 and Y^2 are non-singular in codim 1 (so they are projectively normal). According to this theorem, for said $y = \tau(x)$ to be a point at Y^2 for $x \in X^2$ can only happen in one of two situations:

(1) if τ^{-1} is regular at y, or

(2) if there is a divisor $\mathcal{D}$ on $\mathcal{F}$, $x \in \mathcal{D}$ (called an exceptional divisor), the projective closure $\overline{\tau(\mathcal{D})}$ has dimension 0.

An irreducible curve on $\mathcal{F} \cap \{N \neq 0\}$ can therefore be the pre-image either of a curve or of a point on $\mathcal{G} \cap \{N_1 \neq 0\}$, so the hypothesis in Proposition 1 is necessary. Now, we have to demonstrate that it is also sufficient.

According to the hypothesis in Proposition 1, every irreducible curve $\mathcal{D}'$ on the set of non-regularity for τ is a set-theoretic complete intersection of $\mathcal{F}$ with a surface $\mathcal{R}'$ in $\mathbf{P}^3$. To show that $\mathcal{F}$ is almost-factorial, we need to verify that every irreducible curve $\mathcal{D}$ on $\mathcal{F} \cap \{N \neq 0\}$, whose image is a curve on $\mathcal{G}$, $\mathcal{C} = \tau(\mathcal{D})$ is actually a set-theoretic complete intersection of $\mathcal{F}$.

Denoting the restrictions of τ as τ'', τ', we have the following situation:

$$\begin{array}{ccccc} \mathcal{D} & \subset & \mathcal{F} & \subset & \mathbf{P}^3 \\ \downarrow \tau'' & & \downarrow \tau' & & \downarrow \tau \, . \\ \mathcal{C} & \subset & \mathcal{G} & \subset & \mathbf{P}_1^3 \end{array}$$

As $\mathcal{G}$ is almost-factorial, a surface $\mathcal{H} : H(T_1, X_1, Y_1, Z_1) = 0$ in $\mathbf{P}_1^3$ exists such that $\mathcal{H} \cap \mathcal{G} = \mathcal{C} = \tau(\mathcal{D})$. Let us consider the polynomial $H(H_0, H_1, H_2, H_3) = L(T, X, Y, Z)$ and denote the surface $\mathcal{L} : L(T, X, Y, Z) = 0$.

We consider the divisor

$$\mathcal{F} \cdot \mathcal{L} = \nu\mathcal{D} + \nu_1\mathcal{D}'_1 + \ldots + \nu_t\mathcal{D}'_t, \quad \nu > 0, \, \nu_i > 0, 1 \le j \le t.$$

None of the components $\mathcal{D}'_j \neq \mathcal{D}, 1 \le j \le t$, can be transformed in $\mathcal{C}$ because τ is invertible on $\mathcal{C} = \tau(\mathcal{D})$, so it belongs to $\mathcal{F} \cap \{N = 0\}$, and $\tau(\mathcal{D}'_j)$ is a point on $\mathcal{G}$.

Based on the hypothesis in Proposition 1, there are suitable surfaces $\mathcal{R}_j : R_j(T, X, Y, Z) = 0, 1 \le j \le t$, that cut on $\mathcal{F}$ the divisors

$$\mathcal{R}_j \cdot \mathcal{F} = \mu_j\mathcal{D}'_j, \quad \mu_j > 0, \quad 1 \le j \le t.$$

Let $\mu = m.c.m.\{\mu_1, \ldots, \mu_t\}$ and $R = \prod_{i=1}^{t} R_j^{\frac{\mu\nu_j}{\mu_j}}$. If we consider the surface $\mathcal{R} : R = 0$ in $\mathbf{P}^3$, we have first

$$\mathcal{F} \cdot \mathcal{R} = \mu(\nu_1\mathcal{D}'_1 + \ldots + \nu_t\mathcal{D}'_t),$$

and afterwards the canonical projections of L and of R in $k[\mathcal{F}]$, called l and r, provide

$$\operatorname{div}\left(\frac{l^\mu}{r}\right) = \mu(\nu\mathcal{D} + \nu_1\mathcal{D}'_1 + \ldots + \nu_t\mathcal{D}'_t) - \mu(\nu_1\mathcal{D}'_1 + \ldots + \nu_t\mathcal{D}'_t) = \mu\nu\mathcal{D}.$$

We can apply the theorem of the integral closed Nœtherian's domain to $k[\mathcal{F}]$, and this enables us to find a polynomial $S \in k[T, X, Y, Z]$ such that it defines the divisor $\mu\nu\mathcal{D}$ on $\mathcal{F}$. The surface $\mathcal{S} : S = 0$, thus intersects $\mathcal{F}$ along the curve $\mathcal{D}$ with multiplicity $\mu\nu$. $\mathcal{D}$ is therefore a set-theoretic complete intersection of $\mathcal{F}$, and $\mathcal{F}$ is therefore almost-factorial.

Lemma 1. *If a pair of skew curves $\mathcal{C}_1$ and $\mathcal{C}_2$ exists on a surface $\mathcal{F}$ of $\mathbf{P}^3$, then $\mathcal{F}$ is not almost-factorial.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two skew curves on $\mathcal{F}$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. If $\mathcal{F}$ is almost-factorial, then there are two surfaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathbf{P}^3$ such that

$$\mathcal{F} \cdot \mathcal{H}_1 = n_1\mathcal{C}_1, \quad \mathcal{F} \cdot \mathcal{H}_2 = n_2\mathcal{C}_2.$$

As every curve in $\mathbf{P}^3$ has a non-empty intersection with every surface, we have the contradiction

$$\emptyset \neq \mathcal{C}_1 \cap \mathcal{H}_2 = \mathcal{F} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{C}_1 \cap \mathcal{C}_2.$$

Remark 1. The proof of Proposition 1 enables us to construct a suitable surface $\mathcal{S}$ for which, given an almost-factorial surface $\mathcal{F}$ and an irreducible curve thereon $\mathcal{D}$, it holds that $\mathcal{D} = \mathcal{F} \cap \mathcal{S}$. In the case of $\mathcal{F}$ being rational, see the example in [6].

Lemma 2. *Let $\mathcal{F}$ be a normal surface in $\mathbf{A}^3$, $\deg \mathcal{F} = n \geq 2$ and $\mathbf{r}$ be a straight line on $\mathcal{F}$. For $\mathbf{r}$ to be a complete intersection of $\mathcal{F}$ with a surface $\mathcal{G}$,*

$$\mathcal{F} \cdot \mathcal{G} = \mu\mathbf{r}, \quad \mu = \deg \mathcal{F} \deg \mathcal{G},$$

it is necessary that the plane tangent to $\mathcal{F}$ along $\mathbf{r}$ remains fixed.

Proof. This follows from the statement proved in [7].

3. The Quartic Surfaces in $\mathbf{P}^3$ with a Tacnodal Point

Below, $\mathcal{F}$ will be a normal quartic algebraic surface in $\mathbf{P}^3$ with a tacnodal point, that we can assume to be $(0:0:0:1) = Z_\infty$, and we can take the cone tangent to $\mathcal{F}$ at it (such a cone consists of two coincident planes) to be $T = 0$ (called the *tacnodal tangent plane*). $\mathcal{F}$ will then be given by the equation:

$$Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0, \quad \phi_4, \phi_2 \in k[T, X, Y],$$

where ϕ_4 and ϕ_2 are homogeneous polynomials, ϕ_4 of degree four and ϕ_2 of degree two, or $\phi_2 = 0$. Let us denote $\Delta = \phi_2^2 - \phi_4$, then the surface $\mathcal{F}$ can be represented again with an equation in the form

$$\mathcal{F} : (ZT + \phi_2)^2 - \Delta = 0.$$

We call $T = 0$, the *plane to infinity* of the affine space $\mathbf{A}^3 = \mathbf{P}^3 - \{T = 0\}$.

To recognize and classify the kinds of quartic almost-factorial surfaces with a tacnodal point, we consider the following obvious facts: The section between $\mathcal{F}$ and its tacnodal tangent plane $T = 0$, i.e., $\mathcal{F}_\infty = \mathcal{F} \cap \{T = 0\}$, is splitting into no more than four distinct lines; the cone $\Delta = 0$ of vertex Z_∞ is invariant under linear transformation of coordinates in $\mathbf{A}^3 = \mathbf{P}^3 - \{T = 0\}$, leaving the plane $T = 0$ and the point Z_∞ unchanged.

Lemma 3. *Let* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ *be a quartic surface in* $\mathbf{P}^3$. *Then*

(1) *if two linear polynomials* $\phi_1 = aX + bY + cT$, $\psi_1 = a'X + b'Y + c'T$, *exist with* $ab' - a'b \neq 0$, *such that both divide* ϕ_4, *then a pair of skew lines exists on* $\mathcal{F}$;

(2) *if a linear polynomial* $\phi_1 = aX + bY + cT$, *divides* ϕ_2, *and* ϕ_1^2 *divides* ϕ_4, *then* $\mathcal{F}$ *is singular along the line* $\{T = \phi_1 = 0\}$;

(3) *a surface* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ *has tacnodal points at* O *and at* Z_∞, *if and only if the polynomials* ϕ_2, ϕ_4 *are in* $k[X, Y]$;

(4) *if a surface* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ *has tacnodal points at* O *and at* Z_∞ *and their tangent tacnodal planes intersect at least in two distinct points* A_∞ *and* B_∞ *of* $\mathcal{F}_\infty$, *then* $\mathcal{F}$ *has the skew lines* $Z_\infty A_\infty$, OB_∞;

(5) *on the surface* $\mathcal{F} : Z^2T^2 + 2ZT(aX^2 + 2bXY + cY^2) + X^4 = 0$ *if it is* $c = 0$, *then* $\mathcal{F}$ *is singular along the line* $\{X = T = 0\}$. *Assuming that* $c = 1$, *to within a substitution* $Y = -bX + Y_1$, *we can rewrite the equation for* $\mathcal{F}$ *in the form* $Z^2T^2 + 2ZT[(a - b^2)X^2 + Y_1^2] + X^4 = 0$. *For* $a = b^2 \pm 1$, $\mathcal{F}$ *is singular along the curves* $\{Y_1 = 0 = ZT \pm X^2\}$.

Proof. (1) The lines $\{Z = 0 = aX + bY + cT\}$ and $\{T = 0 = a'X + b'Y + c'T\}$ exist on $\mathcal{F}$. From $ab' - a'b \neq 0$, their intersection is $\{T = Z = X = Y = 0\} = \emptyset$.

(2) The section of $\mathcal{F}$ and the pencil of the planes $\{\phi_1 - \lambda T = 0\}$ is given by

$$Z^2T^2 + 2ZT\lambda T\phi_1' + (\lambda T)^2\phi_2' = T^2(Z^2 + 2Z\phi_1' + \lambda^2\phi_2') = 0,$$

with suitable $\phi_1', \phi_2' \in k[T, X, Y]$.

We obtain the line $\{T = 0 = \phi_1\}$ counted twice for every $\lambda \in k$. $\mathcal{F}$ is then singular along such a line.

(3) $\mathcal{F} : Z^2T^2 + 2ZT\phi_2(X, Y) + \phi_4(X, Y) = 0$ remains invariable under the symmetry of $\mathbf{P}^3$, which changes Z with T. The symmetric of the tacnodal point Z_∞ is the point O, which is then a tacnodal point on $\mathcal{F}$,

and the plane $Z = 0$ is the tacnodal plane tangent to $\mathcal{F}$ at O. Conversely, if O is another tacnodal point of $\mathcal{F}$, we can assume that the tangent tacnodal plane will be $Z = 0$. The equation for $\mathcal{F}$ will be unchanged.

(4) $Z_\infty A_\infty \cap OB_\infty = \emptyset$.

(5) If $c = 0$, then X divides $aX^2 + 2bXY$ and X^2 divides X^4. Then, from (2), $\mathcal{F}$ is singular along $\{X = T = 0\}$. We can assume that $c = 1$.

Substituting $Y = -bX + Y_1$ in $Z^2T^2 + 2ZT(aX^2 + 2bXY + Y^2) + X^4 = 0$, we obtain $Z^2T^2 + 2ZT[(a - b^2)X^2 + Y_1^2] + X^4 = 0$; if $a - b^2 = \pm 1$, then $\mathcal{F} : (ZT \pm X^2)^2 + 2ZTY_1^2 = 0$. At every point on the curves $\{Y_1 = 0 = ZT \pm X^2\}$, the four partial derivatives of $(ZT \pm X^2)^2 + 2ZTY_1^2$ become zero. This shows that $\mathcal{F}$ is singular along these curves.

Proposition 2. *A quartic surface in* $\mathbf{P}_k^3$ *that is non-singular in codimension* 1 *with two tacnodal points is not almost-factorial*, *if the field* k *is supposed more than numerable*.

Proof. Based on Lemma 3, points (4) and (5), we can assume that

$$\mathcal{F} : Z^2T^2 + 2ZT(aX^2 + Y^2) + X^4 = 0, \quad a^2 \neq 1.$$

Let $\sigma : (T_1 : X_1 : Y_1 : Z_1) = (TX^2 : X^3 : TXY : T^2Z)$ be the composition of two blow-ups, one centered in the tacnodal point O of $\mathcal{F}$, and one on a line infinitely near to O. The restriction on $\mathcal{F}$ of σ has as inverse the rational transformation

$$\sigma^{-1} : (T : X : Y : Z) = (T_1^3 : T_1^2X_1 : T_1X_1Y_1 : X_1^2Z_1).$$

The proper transform by σ of $\mathcal{F}$ is the cubic surface

$$\mathcal{G} : Z_1^2T_1 + 2Z_1(aT_1^2 + Y_1^2) + T_1^3 = 0,$$

$\mathcal{G}$ is a non-singular cone in codimension 1 (elliptic), if $a^2 \neq 1$. So $\mathcal{G}$ and $\mathcal{F}$ are birationally equivalent non-rational surfaces. The locus of the non-regularity of σ is given by $M = 0$, where $M = T^2X^6$, because

$$\sigma \circ \sigma^{-1} : (T : X : Y : Z) = (T^3X^6 : T^2X^7 : T^2X^6Y : T^2X^6Z).$$

The section of $\mathcal{F}$ with the set of non-regularity of σ is

$$\mathcal{F} \cap \{XT = 0\} = \{X = Z = 0\} \cup \{X = T = 0\} \cup \{X = ZT + 2Y^2 = 0\}.$$

The lines $r : \{X = Z = 0\}$, $s : \{X = T = 0\}$ are the complete sections of $\mathcal{F}$ with the tacnodal planes tangent to $\mathcal{F}$ at O and at Z_∞.

Now, we show that the conic $\mathcal{C} : \{X = ZT + 2Y^2 = 0\}$ is a set-theoretic complete intersection of $\mathcal{F}$. $\mathcal{F}$ is normal, so $\operatorname{div}\left(\frac{-x^4}{zt}\right)$ on it is

$$4\mathcal{C} + 4\mathbf{r} + 4\mathbf{s} - 4\mathbf{r} - 4\mathbf{s} = 4\mathcal{C} = \operatorname{div}\left(\frac{z^2t^2 + 2zt(ax^2 + y^2)}{zt}\right)$$

$$= \operatorname{div}(zt + 2(ax^2 + y^2)).$$

The surface $\{ZT + 2(aX^2 + Y^2) = 0\}$ thus intersects $\mathcal{F}$ along $4\mathcal{C}$.

So, from Proposition 1, $\mathcal{F}$ is almost-factorial if and only if $\mathcal{G}$ is almost-factorial.

We know that $\mathcal{G}$ is not almost-factorial (see [13], Proposition 11, page 171, where the field k is supposed more than numerable), so $\mathcal{F}$ will not be either.

Lemma 4. *Let* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$, $\phi_4, \phi_2 \in k[T, X, Y]$, *be a normal quartic almost-factorial surface. Then, the plane tangent to* $\mathcal{F}$ *along every component line of* $\mathcal{F}_\infty$ *can only be* $T = 0$.

In addition, only one of the following cases can happen:

(1) $\mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}$, $\mathbf{r}, \mathbf{s}$ *straight lines*, $\mathbf{r} \neq \mathbf{s}$;

(2) $\mathcal{F}_\infty = 4\mathbf{r}$.

If T divides ϕ_2, *then only case* (2) *holds.*

Proof. Let $\mathbf{r} \cup \mathbf{s} \subset \mathcal{F}_\infty$ with $\mathbf{r} \neq \mathbf{s}$. We can assume, for example, that $\mathbf{r} : T = Y = 0$, $\mathbf{s} : T = X + aY = 0$, $a \in k$. From Lemma 2, it follows that the plane tangent to $\mathcal{F}$ along $\mathbf{r}$ remains fixed; if it were different from $T = 0$, let us assume, for instance, that $\alpha : Y = 0$. From $I(\mathbf{r}, \mathcal{F} \cap \alpha) \geq 2$, there must therefore be suitable homogeneous polynomials $\phi_1', \phi_1'', \phi_3', \phi_2' \in k[T, X, Y]$, such that, $\phi_2 = Y\phi_1' + T\phi_1''$, $\phi_4 = Y\phi_3' + T^2\phi_2'$, resulting in

$$\mathcal{F} : T^2(Z^2 + 2Z\phi_1'' + \phi_2') + Y(2ZT\phi_1' + \phi_3') = 0.$$

The section between $\mathcal{F}$ and α is

$$\mathcal{F} \cdot \{Y = 0\} = 2\mathbf{r} + \mathcal{C}, \text{ where } \mathcal{C} : \{Y = Z^2 + 2Z\phi_1'' + \phi_2' = 0\}.$$

The two curves $\mathbf{s}$ and $\mathcal{C}$ on $\mathcal{F}$ are skew because their intersection is

$$\mathbf{s} \cap \mathcal{C} = \{T = X + aY = Y = Z^2 + 2Z\phi_1'' + \phi_2' = 0\} = \emptyset.$$

From Lemma 1, this contradicts the assumption that $\mathcal{F}$ is almost-factorial.

$\mathcal{F}_\infty$ thus consists of two distinct lines at most. Along these lines, the (fixed) plane tangent to $\mathcal{F}$ is $T = 0$. So we can only have the two situations

$$\mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}, \quad \mathbf{r}, \mathbf{s} \text{ straight lines, } \mathbf{r} \neq \mathbf{s}, \text{ or } \mathcal{F}_\infty = 4\mathbf{r}.$$

Now, let us suppose that $\mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}$, $\mathbf{r}, \mathbf{s}$ distinct lines, and $\phi_2 = T\phi_1$, with $\phi_1 \in k[T, X, Y]$. Then $\mathcal{F} : Z^2T^2 + 2ZT^2\phi_1 + \phi_4 = 0$. To within a linear change of coordinates in $\mathbf{P}^3$, we can assume that $\mathcal{F}_\infty : \{X^2Y^2 = 0 = T\}$ and let

$$\phi_4 = X^2Y^2 + T(aX^3 + bX^2Y + cXY^2 + dY^3) + T^2\psi_2, \quad \psi_2 \in k[T, X, Y].$$

The equation for the surface $\mathcal{F} : Z^2T^2 + 2ZT^2\phi_1 + \phi_4 = 0$, can also be written in the form

$$\mathcal{F} : [Y^2 + T(aX + bY)][X^2 + T(cX + dY)]$$
$$+ T^2[Z^2 + 2Z\phi_1 - (aX + bY)(cX + dY) + \psi_2] = 0.$$

This means that the line $\mathbf{r} : T = X = 0$ and the quartic $\mathcal{Q}$, which is the intersection of the quadrics $Y^2 + T(aX + bY) = 0$ and $Z^2 + 2ZT\phi_1 - (aX + bY)(cX + dY) + \psi_2 = 0$, belong to $\mathcal{F}$. Now, the curves $\mathbf{r}$ and $\mathcal{Q}$ on $\mathcal{F}$ are skew because

$$\mathbf{r} \cap \mathcal{Q} = \{T = X = Y^2 = Z^2 = 0\} = \emptyset.$$

This fact, from Lemma 1, contradicts the hypothesis that $\mathcal{F}$ is almost-factorial. So, if T divides ϕ_2, then $\mathcal{F}_\infty = 4\mathbf{r}$.

4. Quartic Surfaces in $\mathbf{P}^3$ with a Tacnodal Point and Two Distinct Principal Tangents

As a first step in the investigation into almost-factoriality for the quartic surfaces in $\mathbf{P}^3$ with a tacnodal point, we have

Lemma 5. *Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ be a quartic surface in $\mathbf{P}^3$ with $\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}$, $\mathbf{r} \neq \mathbf{s}$. If we assume that surfaces $\mathcal{G} : G = 0$ and $\mathcal{H} : H = 0$ exist such that $4\deg(\mathcal{G})\mathbf{r} = \mathcal{G} \cdot \mathcal{F}$ and $4\deg(\mathcal{H})\mathbf{s} = \mathcal{H} \cdot \mathcal{F}$, then the surfaces $\mathcal{G}, \mathcal{H}$ must be quadrics $Q_1 = 0$, $Q_2 = 0$ and $\mathcal{F}$ can be written as*

$$\mathcal{F} : Q_1Q_2 + T^4 = 0.$$

Proof. Let us take $\mathcal{F}$, with $\mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}$, $\mathbf{r} \neq \mathbf{s}$. To within a suitable choice of the coordinates in $\mathbf{P}^3$, we can assume that $\mathcal{F}_\infty = \{X^2Y^2 = 0\} \cdot \{T = 0\}$, where $\mathbf{r} : X = T = 0$, $\mathbf{s} : Y = T = 0$.

Let $\mathcal{G} : G = 0$ and $\mathcal{H} : H = 0$ be surfaces of degree n such that $4n\mathbf{r} = \mathcal{G} \cdot \mathcal{F}$ and $4n\mathbf{s} = \mathcal{H} \cdot \mathcal{F}$ (we can assume that $\deg(G) = \deg(H)$, substituting possible G and H with their suitable powers). In the pencil of surfaces $\Phi : \lambda GH + \mu T^{2n} = 0$, we consider $\Phi_0 : \lambda_0 GH + \mu_0 T^{2n} = 0$, which passes through a point P_0 on $\mathcal{F} \cap \{T \neq 0\}$. The surface Φ_0 cut on $\mathcal{F} : F = 0$ a divisor that is the sum of $4n\mathbf{r} + 4n\mathbf{s}$ and of a curve passing through P_0, and its degree is strictly greater than $8n = \deg \Phi_0 \deg F$. According to Bézout's theorem, the surface $\Phi_0 = 0$ is reducible and $\mathcal{F}$ is therefore one of its components. So, a homogeneous polynomial $L \in k[T, X, Y]$ of degree $2n - 4$ exists for which

$$\lambda_0 GH + \mu_0 T^{2n} = FL.$$

But the intersection between $T^{2n} = 0$ and $\Phi_0 : \lambda_0 GH + \mu_0 T^{2n} = 0$ is $4n\mathbf{r} + 4n\mathbf{s}$, and this coincides with the intersection between $T^{2n} = 0$ and $\mathcal{F}$. It follows that $\{L = T = 0\} = \emptyset$, thus $L = c \in k, c \neq 0$, $\deg(L) = 8n - 4 = 0$, then $n = 2$. In the light of all the above, $\lambda_0 GH + \mu_0 T^4 = cF$ and we can assume $\lambda_0 = c, \mu_0 = c$, so we can write $F = Q_1 Q_2 + T^4$, where

$$Q_1 = X^2 + T(Z + aX + bY + cT) = 0,$$

and

$$Q_2 = Y^2 + T(Z + a'X + b'Y + c'T) = 0.$$

The coefficients of the monomials X^2 and Y^2 can both be assumed to be 1.

Instead of $Q_2 = Y^2 + T(Z + a'X + b'Y + c'T) = 0$, we can take $Q_2 = Y^2 + TZ = 0$ by substituting Z with $Z - a'X - b'Y - c'T$ and (a, b, c) with $(a - a', b - b', c - c')$.

Proposition 3. *The quartics* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$ *in* $\mathbf{P}^3$ *with* $\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 2\mathbf{r} + 2\mathbf{s}$, $\mathbf{r} \neq \mathbf{s}$, *are almost-factorial if and only if they can be written in the form*

$$\mathcal{F}(1) : [X^2 + T(Z + aX + bY + cT)][Y^2 + TZ] + T^4 = 0,$$

$$\textit{with } (4c + b^2 - a^2)^2 = 64.$$

$\mathcal{F}(1)$ *are* 8-*almost-factorial.*

Proof. From Lemma 5, we can assume that $\mathcal{F} : Q_1Q_2 + T^4 = 0$, where $Q_1 = X^2 + T(Z + aX + bY + cT) = 0$ and $Q_2 = Y^2 + TZ = 0$.

Each of the straight lines $\mathbf{r}$, $\mathbf{s}$ is a complete intersection of $\mathcal{F}$ with multiplicity 8 because

$$\mathcal{F} \cdot \{Q_1 = 0\} = 8\{T = X = 0\} \text{ and } \mathcal{F} \cdot \{Q_2 = 0\} = 8\{T = Y = 0\}.$$

In addition, we obtain a parametrization of $\mathcal{F}$ on $\mathbf{P}^2$ by means of

$$\tau : (T : X : Y : Z) = (WPP_1 : WPP_2 : P_3 : WP^2),$$

with the following polynomials of $k[W, U, V]$:

$$P = UW(2V - aW - bW),$$

$$P_1 = -UV^2 - U^2W + bUVW + cUW^2 - W^3,$$

$$P_2 = UV^2 - U^2W - aUVW + cUW^2 - W^3,$$

$$P_3 = UP^2 - WP_1^2 - PW(aP_1 + bP_2 + cP).$$

On $\mathcal{F}$ the transformation

$$(W : U : V) = (T^2 : (X^2 + ZT + T(aX + bY + cT)) : (Y - X)T),$$

is the inverse of τ. To apply Proposition 1 to $\mathcal{F}$, we must compute the polynomial defining the set of non-regularity of the parametrization on $\mathcal{F}$. In the present case, the polynomial is

$$M = T^7[(a+b)T + 2X - 2Y]^2[X^2 + TZ + T(aX + bY + cT)]^2,$$

$M = 0$ intersects $\mathcal{F}$ along the two lines $\mathbf{r}$, $\mathbf{s}$, and along the section between $\mathcal{F}$ with the plane

$$\pi : (a+b)T + 2X - 2Y = 0.$$

As a result, the intersection between π and $\mathcal{F}$ generically splits into two conics, $\mathcal{C}_1$ and $\mathcal{C}_2$, and they coincide if $(4c + b^2 - a^2)^2 = 64$. For $c = \frac{a^2 - b^2}{4} - 2$ and for $c = \frac{a^2 - b^2}{4} + 2$, indeed we have, respectively, the surfaces

$$\left(\frac{a+b}{2}T + X\right)^2 - T^2 + TZ = 0, \quad \left(\frac{a+b}{2}T + X\right)^2 + T^2 + TZ = 0.$$

On these two surfaces, the plane $\pi : (a+b)T + 2X - 2Y = 0$ intersects them along a conic counted twice. We denote with

$$\mathcal{F}(1) : [X^2 + T(Z + aX + bY + cT)][Y^2 + TZ] + T^4 = 0,$$

$$\text{with } (4c + b^2 - a^2)^2 = 64.$$

$\mathcal{F}(1)$ is almost-factorial and its index of almost-factoriality is $\nu = 8$.

Now, we prove that $\mathcal{F}$ is not almost-factorial if $(4c + b^2 - a^2)^2 \neq 64$.

Let $d = b - a$. When c satisfies the equation $2cd^2 + d^3b + 8 = 0$, the irreducible conics

$$\mathcal{D} : \begin{cases} 2X + 2Y - dT &= 0, \\ ZT + X^2 - dXT &= 0, \end{cases}$$

are a subset of $\mathcal{F}$. We consider the affine space $\mathbf{A}^3 = \mathbf{P}^3 \cap \{T \neq 0\}$ and we will have $\mathcal{F}_a = \mathcal{F} \cap \mathbf{A}^3$ and $\mathcal{D}_a = \mathcal{D} \cap \mathbf{A}^3$. As $\mathcal{D}(\mathcal{D}_a)$ are irreducible curves, if $\mathcal{D}$ is a set-theoretic complete intersection of $\mathcal{F}$, then $\mathcal{D}_a$ will be a set-theoretic complete intersection of $\mathcal{F}_a$ too.

Let $x = \frac{X}{T}$, $y = \frac{Y}{T}$, $z = \frac{Z}{T}$ be affine coordinates in $\mathbf{A}^3$. We have

$$\mathcal{D}_a = \begin{cases} y = -x + \frac{d}{2}, \\ z = -x^2 + dx. \end{cases}$$

Now, we consider De Jonquieres' transformation of $\mathbf{A}^3$

$$DJ : \{x_1 = x,\ y_1 = y + x_1 - \frac{d}{2},\ z_1 = z - dx_1 + x_1^2\},$$

DJ transform $\mathcal{F}_a$ and $\mathcal{D}_a$, respectively, on a surface $\mathcal{G}_a$ and on the straight line $\mathbf{r} : \{y_1 = z_1 = 0\}$ on $\mathcal{G}_a$. As DJ is an isomorphism of $\mathbf{A}^3$, $\mathcal{D}_a$ is a complete intersection of $\mathcal{F}_a$ if $\mathbf{r}$ is a complete intersection of $\mathcal{G}_a$. This fact holds if it is satisfied the necessary condition stated in [7] (Lemma 2) analyzing how varies the plane tangent to $\mathcal{G}_a$ at a generic point $(p, 0, 0) \in \mathbf{r}$. This is $(d^5 + ad^4 - 16d + 32p)y_1 + (d^4 - 16)z_1 = 0$. As $(p, 0, 0)$ moves along $\mathbf{r}$, the plane remains fixed if and only if $d^4 - 16 = 0$, and it is only in this case that the straight line $\mathbf{r}$ can be a complete intersection on the surface $\mathcal{G}_a$.

Now we compute $A = (4c + b^2 - a^2)^2$, substituting $b = d + a$, $c = -\frac{d^3 b + 8}{2d^2}$ in A. Thus, $A = 32 + \frac{256}{d^4} + d^4$ and $A = 64$ if and only if $d^4 = 16$. This leads us to conclude that, if $(4c + b^2 - a^2)^2 \neq 64$, then the curves $\mathcal{D}$ on $\mathcal{F}$ cannot be a set-theoretic complete intersection of $\mathcal{F}$. This goes to show that $\mathcal{F}$ is almost-factorial if and only if $(4c + b^2 - a^2)^2 = 64$.

5. Quartic Surfaces in $\mathbf{P}^3$ with a Tacnodal Point and Only One Principal Tangent

We can write $\mathcal{F} : (ZT + \phi_2)^2 - \Delta = 0$, where $\Delta = \phi_2^2 - \phi_4$.

Let us consider the birational transformation $\tau : \mathbf{P}^3 \longrightarrow \mathbf{P}_1^3$

$$\tau : (T_1 : X_1 : Y_1 : Z_1) = (T^2 : TX : TY : (TZ + \phi_2)), \tag{4}$$

and we have, for its restriction on $\mathcal{F}$

$$\tau^{-1} : (T : X : Y : Z) = (T_1^2 : T_1X_1 : T_1Y_1 : (T_1Z_1 - \phi_2')),$$

where $\phi_2' = \phi_2(T_1, X_1, Y_1)$.

The set of non-regularity of τ is $T = 0$, and $T_1 = 0$ for τ^{-1}, because we have

$$\tau \circ \tau^{-1} : (T : X : Y : Z) = (T^4 : T^3X : T^3Y : T^3Z),$$

and

$$\tau^{-1} \circ \tau : (T_1 : X_1 : Y_1 : Z_1) = (T_1^4 : T_1^3X_1 : T_1^3Y_1 : T_1^3Z_1).$$

Using τ, we obtain

$$\tau(\mathcal{F}) : [T_1Z_1 - \phi_2(T_1, X_1, Y_1)]T_1^2 + \phi_2(T_1^2, X_1T_1, Y_1T_1)\}^2 - \Delta(T_1^2, X_1T_1, Y_1T_1)$$

$$= T_1^4[Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1)] = 0.$$

The proper transform of $\mathcal{F}$ by τ is $\mathcal{H} : Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1) = 0$.

If $\mathcal{F}$ is almost-factorial, then the affine part $\mathcal{F} \cap \{T \neq 0\}$ of $\mathcal{F}$ is almost-factorial too because the exceptional divisor for τ is the line $\mathcal{F} \cap \{T = 0\}$, counted 4 times. So $\mathcal{H}$ is almost-factorial if its affine part $\mathcal{H} \cap \{T_1 \neq 0\}$ is almost-factorial, and $\mathcal{H}_\infty = \mathcal{H} \cap \{T_1 = 0\}$ is a complete intersection in $\mathbf{P}_1^3$.

To examine $\mathcal{H}$, we have to distinguish between three cases where $\Delta = \phi_2^2 - \phi_4$ is irreducible, or a factor of Δ is T or not T.

5.1. The surfaces $\mathcal{F}$ with $\mathcal{F}_\infty = 4r$ and Δ are irreducible

First, we examine a surface of the equation $\mathcal{F} : Z^2T^2 + \phi_4(T, X, Y) = 0$ of the kind $\mathcal{H} : Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1) = 0$ just found.

Lemma 6. *The polynomial* $\phi_4 = -X^4 + TY^3 + T(X\psi_2 + T\chi_2)$ *is irreducible for every homogeneous polynomial* $\psi_2, \chi_2 \in k[T, X, Y]$.

Proof. A suitable splitting of ϕ_4 in $k[T, X, Y]$ would be one of two kinds:

$$\phi_4 = [-X^3 + aX^2Y + bXY^2 + cY^3 + T(\cdots)](X + dY + eT); \quad a, b, c, d, e \in k,$$

$$\phi_4 = [-X^2 + aXY + bY^2 + T(\cdots)][X^2 + cXY + dY^2 + T(\cdots)]; \quad a, b, c, d \in k.$$

As in ϕ_4, X^3Y, X^2Y^2, XY^3 have zero as a coefficient, and in both the equalities it must hold that $a = b = c = d = 0$; but then the monomial TY^3 has zero as a coefficient: Contradiction!. This shows that ϕ_4 is irreducible.

Lemma 7. *Let* $\mathcal{F} : Z^2T^2 + \phi_4 = 0 \subset \mathbf{P}^3$ *be a quartic surface, where*

$$\phi_4 = TY^3 + T[X\psi_2(T, X, Y) + T\chi_2(T, X, Y)] + X^4.$$

A suitable plane $Z = lT$ *exists that intersects the surface* $\mathcal{F}$ *along an irreducible curve* $\mathcal{C} : \mathcal{F} \cap \{Z = lT\}$, *which proves to be singular in at least one point* $P_0 \notin \{T = 0\}$.

By means of a suitable choice of coordinates in $\mathbf{A}^3 = \mathbf{P}^3 - \{T = 0\}$, *we can assume* ϕ_4 *of the form*

$$\phi_4 = -(X + aT)^2X^2 + TY(bX^2 + cXY + Y^2 + dXT + eYT),$$

with $a, b, c, d, e \in k$.

Proof. Let us first prove the existence of a section $\mathcal{C} : \{Z - lT = \phi_4 = 0\}$ on $\mathcal{F}$ that is irreducible and it has a singular point P_0 on it that does not belong to the plane $T = 0$. Let $F = l^2T^4 - X^4 + T\phi_3$, where $\phi_3 = Y^3 + X\psi_2(T, X, Y) + T\chi_2(T, X, Y)$; by choosing suitable $l \in k$, the homogeneous system in T, X, Y

$$\begin{cases} \dfrac{\partial F}{\partial T} = & 4l^2T^3 + \phi_3 + T\dfrac{\partial \phi_3}{\partial T} & = 0, \\ \dfrac{\partial F}{\partial X} = & -4X^3 + T\dfrac{\partial \phi_3}{\partial X} & = 0, \\ \dfrac{\partial F}{\partial Y} = & T\dfrac{\partial \phi_3}{\partial Y} = T[3Y^2 + X(\cdots) + T(\cdots)] & = 0, \end{cases}$$

has at least one solution that gives a point $P_0 = (t_0 : x_0 : y_0 : lt_0)$, with $t_0 \neq 0$. Indeed, a possible point with $t_0 = 0\,(0 : x_0 : y_0 : 0)$ singular for $\mathcal{C}$, would satisfy

$$\{0 = T = X = \phi_3 = Z\} = \{0 = T = X = Y^3 = Z\}.$$

So $x_0 = y_0 = 0$ and such a point P_0 does not exist.

Now $\mathcal{C} : \{Z - lT = l^2T^4 - X^4 + T\phi_3 = 0\}$ cannot be reducible because the plane $Z = lT$ is transversal to the cone $l^2T^4 - X^4 + [(Y^3 + X\psi_2(T, X, Y) + T\chi_2(T, X, Y)] = 0$, which is irreducible because, from Lemma 6, the polynomial $-X^4 + TY^3 + T(X\psi_2 + T\chi_2')$ is irreducible, with $\chi_2' = \chi_2 + l^2T^2$.

To within a change of affine coordinates, we can assume that the plane $Z = lT$ is $Z = 0$, that the singular point on $\mathcal{C}$ is $P_0 = O = (1 : 0 : 0 : 0)$, and that one of the lines tangent to $\mathcal{C}$ through P_0 is $Z = Y = 0$ at the point $A = (1 : -a : 0 : 0)$. These assumptions can be drawn in the light of the fact that $\phi_4 = -X^4 + T\phi_3$, and the coefficient of Y^3 in ϕ_3 is 1. In this chosen frame, the curve $\mathcal{C}$ is given by

$$\mathcal{C} : \{Z = 0 = -(X + aT)^2X^2 + TY(bX^2 + cXY + Y^2 + dXT + eYT)\},$$

with $a, b, c, d, e \in k$.

Proposition 4. *The normal quartic surface of* $\mathbf{P}^3$

$$\mathcal{F} : Z^2T^2 + \phi_4 = 0, \quad \phi_4 \in k[T, X, Y], \textit{ with } \mathcal{F}_\infty = 4\mathbf{r},$$

is almost-factorial if and only if, to within a suitable linear change of coordinates, it is

$$\mathcal{F}^* : Z^2T^2 - X^4 + Y^3T = 0.$$

$\mathcal{F}^*$ *is* 12-*almost-factorial.*

Proof. We can assume that $\mathbf{r} = Z_\infty Y_\infty$ and now $\phi_4 = -X^4 + T\phi_3$, where $\phi_3 = Y^3 + (X\psi_2 + T\chi_2)$, and $\psi_2, \chi_2 \in k[T, X, Y]$ are zero or homogeneous polynomials of degree 2. The coefficient of Y^3 can be assumed to be 1 (it cannot vanish otherwise the generic plane $X = \lambda T$ would intersect $\mathcal{F}$ along the lines $T = X = 0$ counted at least twice, in which case $\mathcal{F}$ would be singular in codimension 1). So, the hypotheses of Lemmas 6 and 7 hold, for which we can assume that

$$\mathcal{F} : Z^2T^2 - (X + aT)^2 X^2 + TY(bX^2 + cXY + Y^2 + dXT + eYT) = 0.$$

Now, let us consider the rational transformation $\sigma : \mathbf{P}^3 \longrightarrow \mathbf{P}^3_2$

$$\sigma : (T_2 : X_2 : Y_2 : Z_2) = (T^2 : TX : TY : (ZT - X(X + aT)),$$

which has the following inverse transformation on $\sigma(\mathcal{F})$:

$$\sigma^{-1} : (T : X : Y : Z) = (T_2^2 : T_2X_2 : T_2Y_2 : (Z_2T_2 + X_2(X_2 + aT_2))).$$

As a result,

$$\sigma^{-1} \circ \sigma : (T_2 : X_2 : Y_2 : Z_2) = (T_2^4 : T_2^3X_2 : T_2^3Y_2 : T_2^3Z_2)).$$

From this, the factor of non regularity is $T_2 = 0$, which proves that σ is biregular on the affine part $\mathbf{P}^3_2 \cap \{T_2 \neq 0\}$. By means of σ, the surface $\mathcal{F}$ will be transformed into

$$T_2^5[T_2Z_2^2 + 2aX_2Z_2T_2 + 2X_2^2Z_2 + bX_2^2Y_2 + cX_2Y_2^2 + Y_2^3 + dX_2Y_2T_2 + eY_2^2T_2] = 0.$$

To within the factor T_2^5, we thus find that the proper transform of $\mathcal{F}$ is the cubic (monoid with a double point at $O_1 = (1 : 0 : 0 : 0)$).

$$\mathcal{F}_3 : T_2(Z_2^2 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2) + Y_2^3 + 2X_2^2Z_2 + bX_2^2Y_2 + cX_2Y_2^2 = 0.$$

$Y_2^3 + 2X_2^2Z_2 + bX_2^2Y_2 + cX_2Y_2^2 = 0$ is an irreducible cubic cone that is singular along the line $\{X_2 = Y_2 = 0\}$ and intersects the quadric cone

$$\Gamma : Z_2^2 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2 = 0,$$

along the line $\{Z_2 = Y_2 = 0\}$. As this line is irreducible, it is a set-theoretic complete intersection of $\mathcal{F}_3$. Then we can apply Proposition 1 to the surface $\mathcal{F}_3$, which is therefore almost-factorial and it must be one of the three almost-factorial surfaces (monoids) classified by Stagnaro (see [13], Theorem on page 143). To within a suitable linear change of coordinates in $\mathbf{P}^3$, these monoids are one of the following kinds:

$$\text{(I)} \quad : \quad T_2(Y_2^2 + X_2Z_2) + X_2^3 = 0,$$

$$\text{(II)} \quad : \quad T_2(X_2Y_2) - Z_2^3 = 0,$$

$$\text{(III)} : \quad T_2(X_2^2) + X_2Z_2^2 + Y_2^3 = 0.$$

A priori, the following cases may occur:

(1) If Γ is irreducible $(4ea^2 - d^2 \neq 0)$, then $\mathcal{F}_3$ will be the surface (I) for which the six lines on $\mathcal{F}_3$ passing through O_1 must concide. This condition is met if the resultant polynomial, with respect to Z_2 (which is now $Z_{2\infty} \notin \Gamma$) between

$$Z_2^2 + 2aX_2Z_2 + dX_2Y_2 + eY_2^2 \text{ and } Y_2^3 + 2X_2^2Z_2 + X_2Y_2(bX_2 + cY_2),$$

is a sixth power of one linear form in X_2, Y_2. But such a resultant is

$$\begin{aligned} &Y_2[4(ab-d)X_2^5 + (4ac - b^2 - 4e)X_2^4Y_2 \\ &\qquad + 2(2a - bc)X_2^3Y_2^2 - (c^2 + 2b)X_2^2Y_2^3 - 2cX_2Y_2^4 - Y_2^5], \end{aligned}$$

which would consequently be divisible by Y_2^6. This implies that

$$c = b = a = e = d = 0,$$

and this contradicts $4a^2e - d^2 \neq 0$. So $\mathcal{F}$ cannot be transformed into the $\mathcal{F}_3$ of the kind (I).

(2) The cone Γ is reducible (i.e., $d^2 = 4ea^2$) in two distinct or coincident planes. In the first case, $\mathcal{F}_3$ is of the kind (II) and both of the two component planes of the cone must intersect the monoid along three coincident lines. The resultant must then be divisible first by Y_2^3, and then by the third power of a linear form in X_2, Y_2. This implies that the resultant is Y_2^6, so $b = c = 0$. Γ is therefore the plane $Z_2 = 0$ counted twice. This means that the surface is of the kind (III), with $a = d = e = 0$, and the plane $Z_2 = 0$ must intersect $\mathcal{F}_3$ along a line counted three times. We have $\mathcal{F}_3 : T_2Z_2^2 + Y_2^3 + 2X_2^2Z_2 = 0$. To within a change of coordinates in $\mathbf{P}^3$, the assigned surfaces

$$\mathcal{F} : Z^2T^2 + \phi_4 = 0, \quad \phi_4 \in k[T, X, Y], \text{ with } \mathcal{F}_\infty = 4\mathbf{r},$$

are

$$\mathcal{F}^* : Z^2T^2 - X^4 + TY^3 = 0.$$

This demonstrates Proposition 4. We note that the point $A = (1 : -a : 0 : 0)$ becomes O, and $\mathcal{C}$ has a triple point at O. $\mathcal{F}^*$ can be parametrized on the plane $\mathbf{P}^2$ by

$$(T : X : Y : Z) = (W^3(W^2 - U^2)^2 : W^2V^3(W^2 - U^2) : WV^4(W^2 - U^2) : V^6U).$$

The inverse correspondence on $\mathcal{F}^*$ is $(W : U : V) = (X^2 : TZ : XY)$ with the factor of non-regularity $M = X^6Y^6T$.

It is easy to see that the surface $\mathcal{F}^*$ is 12-almost-factorial. The surface $\mathcal{F}^*$ was investigated in [6], Example 1, page 290.

Remark 2. In the above-mentioned paper [6], it had been shown how computing a surface $\mathcal{G}$, whose complete intersection with $\mathcal{F}$ is $\mathcal{C}$ with multiplicity $\mu = \deg(\mathcal{F})\deg(\mathcal{G})/\deg(\mathcal{C})$ for every irreducible curve $\mathcal{C}$ on $\mathcal{F}$, and some examples were given.

Proposition 5. *Let* $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$, *be a quartic surface on* $\mathbf{P}^3$ *with* $\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 4\mathbf{r}$, *and* $\Delta = \phi_2^2 - \phi_4$ *is irreducible.* $\mathcal{F}$ *is almost-factorial if and only if* $\mathcal{F}$ *can be written in the form*

$$\mathcal{F}(2) : (ZT + aX^2)^2 - X^4 + Y^3T = 0, \quad a^2 \neq 1.$$

$\mathcal{F}$ *will be* 12-*almost-factorial.*

Proof. We have $\mathcal{F} \cdot \{T = 0\} = 4\mathbf{r}$, so we can write

$$\phi_4 = X^4 + T\phi_3(T, X, Y), \quad \phi_3 \in k[T, X, Y].$$

Let $\phi_2 = aX^2 + bXY + cY^2 + T(dX + eY + fT), \quad a, b, c, d, e, f \in k.$

From the birational transformation (4), the proper transform of $\mathcal{F}$ by τ is $\mathcal{H} : Z_1^2T_1^2 - \Delta(T_1, X_1, Y_1) = 0$, where Δ is irreducible. So, we can apply Proposition 4 to $\mathcal{H}$. In particular, in a suitable coordinates' frame of $\mathbf{P}_1^3$, we see that $\Delta' = \phi_2(T_1, X_1, Y_1)^2 - X_1^4 - T\phi_3(T_1, X_1, Y_1)$ can be written in the form of $- X_1^4 + Y_1^3T_1$. Now Y_1^4 and $X_1^2Y_1^2$ do not appear in Δ', so it necessarily is $b = c = 0$, and therefore, we have $\phi_2 = aX^2 + T(dX + eY + fT)$, $d, e, f \in k$. By replacing the coordinates in $\mathbf{P}^3$ with

$$T' = T, \quad X' = X, \quad Y' = Y, \quad Z' = Z + dX + eY + fT,$$

we obtain surfaces of the kind

$$(Z'T' + aX'^2)^2 - X'^4 + Y'^3T' = 0, \quad a^2 \neq 1.$$

We can thus conclude that, to within a change of coordinates, when $\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 4\mathbf{r}$ and $\Delta = \phi_2^2 - \phi_4$ is irreducible, the only quartic almost-factorial surfaces in $\mathbf{P}^3$ are of the kind

$$\mathcal{F}(2) : (ZT + aX^2)^2 - X^4 + Y^3T = 0, \quad a^2 \neq 1.$$

$\mathcal{F}(2)$ are 12-almost-factorial.

5.2. The surfaces $\mathcal{F}$ with $\mathcal{F}_\infty = 4r$, Δ reducible, *T* not factor of Δ

Proposition 6. *Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$, be a quartic surface in $\mathbf{P}^3$ with $\mathcal{F} \cdot \{T = 0\} = \mathcal{F}_\infty = 4\mathbf{r}$ and $\Delta = \phi_2^2 - \phi_4$ reducible, but T does not divide Δ. In a suitable coordinates' frame, $\mathcal{F}$ is almost-factorial if and only if it is of the kind*

$$\mathcal{F}(3) : (ZT + aX^2 + Y^2 + T(a_1X + b_1Y))^2$$

$$- [(a-1)X^2 + Y^2 + 2T(dX + eY)][(a+1)X^2 + Y^2 + 2T(fX + gY)] = 0,$$

where $a^2 \neq 1, (e, g) \neq (0, 0)$, *with* $(a + f^2/g^2)^2 = 1$, *and* $\operatorname{def} g \neq 0$. *$\mathcal{F}(3)$ is 4-almost-factorial.*

Proof. Let $\phi_2 = aX^2 + 2bXY + cY^2 + T\phi_1$, and now $\phi_4 = X^4 + T\phi_3$, with $\phi_1, \phi_3 \in k[T, X, Y]$ suitable polynomials vanishing or of degrees 1 and 3, respectively. If $\phi_1 = 0 = \phi_3$, $\mathcal{F}$ has two tacnodal points and we can apply Lemma 3 and Proposition 2 to $\mathcal{F}$. Such a surface is not almost-factorial. Then we can assume that (ϕ_1, ϕ_3) does not vanish and, as done in Lemma 3, (5), we can assume that $c = 1, b = 0, a^2 \neq 1$. Let $\phi_1 = a_1X + b_1Y + c_1T$. Thus

$$\Delta(T, X, Y) = \phi_2^2 - \phi_4 = (aX^2 + Y^2 + T\phi_1)^2 - X^4 - T\phi_3,$$

and, since Δ is reducible but T does not divide Δ every factor

$$(aX^2 + Y^2)^2 - X^4 = [(a-1)X^2 + Y^2][(a+1)X^2 + Y^2],$$

is different from zero. So, it must be that $\Delta = \psi_2\chi_2$ with non-vanishing suitable polynomials of $k[T, X, Y]$. Here we can assume that $O_1 = (1 : 0 : 0 : 0) \in \mathcal{F}$ and at O_1 satisfies $\phi_1 = \psi_2 = \chi_2 = 0$ (possibly by substituting X, Y with suitable $X' = X + uT$, $Y' = Y + vT$, $u, v \in k$). So, we can suppose $c_1 = 0$, so $\phi_1 = a_1X + b_1Y$, and

$$\psi_2 = (a-1)X^2 + Y^2 + 2T(dX + eY),$$

$$\chi_2 = (a+1)X^2 + Y^2 + 2T(fX + gY).$$

The equation for $\mathcal{F}$ can be written in the form: $(ZT + \phi_2)^2 = \psi_2\chi_2$. We can obtain a parametrization of $\mathcal{F}$ on $\mathbf{P}^2$ by assuming first

$$\frac{V}{W} = \frac{ZT + \phi_2}{\psi_2}, \quad \frac{U}{W} = \frac{Y}{X},$$

and then, from the relation

$$\frac{V^2}{W^2} = \frac{(ZT + \phi_2)^2}{\psi_2^2} = \frac{\chi_2}{\psi_2},$$

we shall have $V^2\psi_2 = W^2\chi_2$. So it must be

$$V^2[(a-1)X^2 + \frac{U^2}{W^2}X^2 + 2T\frac{U}{W}(dX + e\frac{U}{W}X]$$

$$= W^2[(a+1)X^2 + \frac{U^2}{W^2}X^2 + 2T\frac{U}{W}(fX + g\frac{U}{W}X)].$$

Omitting the factor X from the relation above, we shall have

$$\frac{X}{T} = \frac{WN}{D} \text{ and } \frac{Y}{T} = \frac{UN}{D},$$

where

$$N = 2[W^2(fW + gU) - V^2(dW + eU)], \tag{5}$$

$$D = V^2[(a-1)W^2 + U^2] - W^2[(1+a)W^2 + U^2]. \tag{6}$$

Finally, from $W(ZT + \phi_2) = V\psi_2$, we first have

$$ZT = \frac{V\psi_2(T, X, Y) - W\phi_2(T, X, Y)}{W}, \tag{7}$$

$\phi_2(T, X, Y)$ and $\psi_2(T, X, Y)$ being homogeneous of degree 2, we shall have

$$\frac{T^2}{D^2}\phi_2\left(\frac{D}{T}T, \frac{D}{T}X, \frac{D}{T}Y\right) = \frac{T^2}{D^2}\phi_2(D, WN, UN),$$

$$\frac{T^2}{D^2}\psi_2\left(\frac{D}{T}T, \frac{D}{T}X, \frac{D}{T}Y\right) = \frac{T^2}{D^2}\psi_2(D, WN, UN).$$

Finally,

$$\frac{Z}{T} = \frac{P}{WD^2},$$

where we have put

$$P = V\psi_2(D, WN, UN) - W\phi_2(D, WN, UN) \in k[U, V, W], \tag{8}$$

a polynomial of degree 9. Only if it is supposed $N \neq 0$, i.e., $(d, e, f, g) \neq (0, 0, 0, 0)$, we obtain a parametrization for the surface $\mathcal{F}$ given by

$$\frac{X}{T} = \frac{W^2ND}{WD^2}, \quad \frac{Y}{T} = \frac{WUND}{WD^2}, \quad \frac{Z}{T} = \frac{P}{WD^2},$$

and this is a restriction to $\mathcal{F}$ of the rational map $\sigma : \mathbf{P}^2 \longrightarrow \mathbf{P}^3$

$$(T : X : Y : Z) = (WD^2 : W^2ND : WUND : P).$$

The rational transformation $\mathbf{P}^3 \longrightarrow \mathbf{P}^2$

$$(W : U : V) = (X\psi_2 : Y\psi_2 : X(ZT + \phi_2)),$$

induces one birational transformation inverse of σ on $\mathcal{F}$. When we substitute, mod $\mathcal{F}$, respectively,

$$W, U, V \text{ with } X\psi_2, Y\psi_2, X(ZT + \phi_2),$$

we have, of course $W = X\psi_2$, $U = Y\psi_2$, and from (5) and (6),

$$D = TX^2\psi_2^3R, \quad N = TX^2\psi_2^2R, \tag{9}$$

where $R = 2(dX + eY)[(a+1)X^2 + Y^2] - 2(fX + gY)[(a-1)X^2 + Y^2]$.

From (7), (8), and (9) results

$$P = V\psi_2(D, WN, UN) - W\phi_2(D, WN, UN)$$

$$= V\psi_2(TX^2\psi_2^3R, X^3\psi_2^3R, YX^2\psi_2^3R) - W\psi_2(TX^2\psi_2^3R, X^3\psi_2^3R, YX^2\psi_2^3R)$$

$$= X^4\psi_2^6R^2[V\psi_2(T, X, Y) - W\phi_2(T, X, Y)] = X^4\psi_2^6R^2ZTW = ZTX^5\psi_2^7R^2.$$

Therefore, we have for T, X, Y, Z, respectively,

$$T(TX^5\psi_2^7R^2), \quad X(TX^5\psi_2^7R^2), \quad Y(TX^5\psi_2^7R^2), \quad Z(TX^5\psi_2^7R^2).$$

Thus, the factor of non-regularity of σ, mod $\mathcal{F}$, is

$$M = TX^5\psi_2^7R^2,$$

where $R = 2(dX + eY)[(a+1)X^2 + Y^2] - 2(fX + gY)[(a-1)X^2 + Y^2]$.

We note that R splits into three linear forms in X, Y, and R is just the resultant of ψ_2 and χ_2 with respect to T.

Now the surface $M = 0$ cuts the following curves on $\mathcal{F}$:

- the section of $\mathcal{F}$ with the quadric $\psi_2 = 0$; this is the irreducible quartic $\mathcal{C}_4 : \psi_2 = ZT + \phi_2 = 0$ (counted twice), if ψ_2 is irreducible; otherwise, if ψ_2 is reducible, we obtain two conics (counted twice), each of which is the intersection of $ZT + \phi_2 = 0$ with a plane component of $\psi_2 = 0$;

- $T = 0$ gives the line $\mathbf{r} : X = T = 0$ on $\mathcal{F}$ counted 4 times;

- the section between $\mathcal{F}$ and $X = 0$ splits into the line $\mathbf{r}$ and the irreducible (rational) plane cubic

$$\mathcal{C}_3 : \{X = 0 = Z^2T + 2ZY(Y + b_1T) + 2Y^3(b_1 - e - g) + Y^2T(b_1^2 - 4eg)\}.$$

Now $\mathbf{r}$ is a set-theoretic complete intersection of $\mathcal{F}$ with $T = 0$ and also $\mathcal{C}_3$ because on $\mathcal{F}$, we have

$$\operatorname{div}_{\mathcal{F}}(-x^4/t) = 4\mathcal{C}_3 + 4\mathbf{r} - 4\mathbf{r} = 4\mathcal{C}_3.$$

If we set $G = Z^2T + 2Z\phi_2 + \phi_3$ is $\operatorname{div}_{\mathcal{F}}(g) = 4\mathcal{C}_3$ and therefore $\mathcal{G} : \{G = 0\}$ intersects $\mathcal{F}$ along $4\mathcal{C}_3$.

- $R = 0$ gives three planes passing through the line $X = Y = 0$, each of them, different from $X = 0$, $Y = tX$ intersects $\mathcal{F}$ along a reducible quartic if and only if, for $Y = tX$ Δ becomes a square $\overline{\Delta} = \overline{\chi_2}\overline{\psi_2} = A^2X^4$, with $A \in k$. We are interested in the planes $Y = tX$ over which it results $\overline{R} = 0$ and so

$$\begin{aligned} &A^2X^4 \\ &= 4(d - et)(f - gt)X^2T^2 + 4(f - gt)(a - 1 + t^2)X^3T + [(a + t^2)^2 - 1]X^4. \end{aligned}$$

When $t = \frac{f}{g}$ and $A^2 = (a + \frac{f^2}{g^2})^2 - 1 \neq 0$, $\overline{R} = 0$ implies $\frac{f}{g} = \frac{d}{e}$. The plane $fX + gY = 0$ intersects $\mathcal{F}$ according the two different conics

$$\{fX + gY = 0 = ZT + (a_1 - b_1 \frac{f}{g})XT + (\pm A + a + \frac{f^2}{g^2})X^2\}.$$

Arguing similarly the final part of proof of Proposition 3, one shows that these conics can not be complete intersection on $\mathcal{F}$.

When $t = \frac{f}{g} = \frac{d}{e}$ and $A^2 = (a + \frac{f^2}{g^2})^2 - 1 = 0$, i.e., $a = \pm 1 - \frac{f^2}{g^2}$, we find the two double conics

$$2\{\psi_2 = 0 = ZT + \phi_2\}, \text{ and } 2\{\chi_2 = 0 = ZT + \phi_2\}.$$

The plane $dX + eY = 0$, if $fe \neq gd$, intersects $\mathcal{F}$ according complete intersections: A double conic if $a = 1 - \frac{d^2}{e^2}$, or an irreducible quartic.

The same is for the plane $fX + gY = 0$, if $fe \neq gd$. It intersects $\mathcal{F}$ according a double conic if $a = -1 - \frac{f^2}{g^2}$, or an irreducible quartic.

From Proposition 1, $\mathcal{F}$ is almost-factorial if and only if $(a + \frac{f^2}{g^2})^2 = 1$, with $\operatorname{def} g \neq 0$. $\mathcal{F}$ results 4-almost-factorial.

Let us denote the said almost-factorial surface with

$$\mathcal{F}(3) : (ZT + aX^2 + Y^2 + T(a_1X + b_1Y))^2$$

$$- [(a-1)X^2 + Y^2 + 2T(dX + eY)][(a+1)X^2 + Y^2 + 2T(fX + gY)] = 0,$$

with $(a + \frac{f^2}{g^2})^2 = 1$, $\operatorname{def} g \neq 0$.

5.3. Quartic surfaces $\mathcal{F}$ with $\mathcal{F}_\infty = 4r$ and with T dividing Δ

We can write $\Delta = T\psi_3(T, X, Y)$, and not $\Delta = T^2\psi_2(T, X, Y)$, because the surface

$$(ZT + \phi_2)^2 - T^2\psi_2(T, X, Y) = 0,$$

would be singular along $\{T = \phi_2 = 0\}$.

In the case under investigation, we obtain, by τ in (4), from the surface $\mathcal{F} : (ZT + \phi_2)^2 - T\psi_3 = 0$ the cubic surface (double plane)

$$\mathcal{H} : Z_1^2 T_1 - \psi_3(T_1, X_1, Y_1) = 0.$$

If $\mathcal{F}$ is assumed to be almost-factorial, so is $\mathcal{F} \cap \{T \neq 0\}$, and the affine part $\mathcal{H}_a = \mathcal{H} \cap \{T_1 \neq 0\}$ must therefore be almost-factorial. Conversely, if $\mathcal{H}_a$ is almost-factorial, so is $\mathcal{F} \cap \{T \neq 0\}$ because the transformation τ is biregular on $\mathbf{P}^3 \cap \{T \neq 0\}$, and being $\mathcal{F}_\infty = 4\mathbf{r}$, $\mathcal{F}$ is also almost-factorial.

We examine all the possibilities in the following proposition:

Proposition 7. *Let $\mathcal{F} : Z^2T^2 + 2ZT\phi_2 + \phi_4 = 0$, be a normal surface in $\mathbf{P}^3$, with $\mathcal{F}_\infty = 4\mathbf{r}$. If T divides $\Delta(T, X, Y) = \phi_2^2 - \phi_4$, $\mathcal{F}$ is almost-factorial if and only if, to within a linear change of affine coordinates in $\mathbf{P}^3 - \{T = 0\}$, $\mathcal{F}$ is one of the following*:

$\mathcal{F}(4)$: $[ZT + (aX + bY)^2]^2 - TXY(X + Y) = 0, \quad 0 \neq a \neq b \neq 0;$

$\mathcal{F}(5)$: $[ZT + (aX + bY)^2]^2 - TXY(X + T) = 0, \quad a \neq 0, b \neq 0;$

$\mathcal{F}(6)$: $[ZT + (aX + bY)^2]^2 - TX(XY + T^2) = 0, \quad a \neq 0, b \neq 0;$

$\mathcal{F}(7)$: $[ZT + (aX + bY)^2]^2 - TX(TY + X^2) = 0, \quad b \neq 0;$

$$\mathcal{F}(8) \quad : \quad [ZT + (aX + bY)^2]^2 - T(T^2Y + X^2) = 0, \quad b \neq 0;$$

$$\mathcal{F}(9) \quad : \quad [ZT + (aX + bY)^2]^2 - T(TY^2 + X^3) = 0, \quad b \neq 0.$$

The surface $\mathcal{F}(9)$ *is* 12-*almost-factorial*; *all the others are* 4-*almost-factorial.*

Proof. As T is a factor of Δ (reducible), in every case it must hold that

$$\mathcal{F}_\infty : \{T = \phi_4 = 0\} = \{T = \phi_2^2 - \Delta = 0\} = 2\{T = \phi_2 = 0\} = 4\mathbf{r},$$

with $\mathbf{r}$ a suitable line, that we can assume $\mathbf{r} : \{T = aX + bY = 0\}$, with a, b satisfying the condition to avoid $\mathcal{F}$ is singular along $\mathbf{r}$. So we have

$$\phi_2 = (aX + bY)^2 + T\phi_1, \quad \Delta = T\psi_3,$$

Δ being reducible we have to examine the case in which the almost-factorial surface $\mathcal{F}$ will be transformed into the double plane

$$\mathcal{H} : Z_1^2 T_1 - \psi_3(T_1, X_1, Y_1) = 0.$$

The exceptional divisor of τ is $\{T_1 = \psi_3(T_1, X_1, Y_1) = 0\}$. This divisor splits into straight lines passing through Z_∞ and its affine part $\mathcal{H}_a = \mathcal{H} \cap \{T_1 \neq 0\}$ must be almost-factorial. The affine cubic surface $\mathcal{H}_a$ is then among those listed in [4]. We are interested only in those that have the projective closure with $\mathcal{H}_\infty$ splitting into lines passing through Z_∞:

(1) they are three distinct lines;

(2) two of the three lines coincide; or

(3) all three lines coincide.

The double cubic planes with the above said property are the following:

For **case 1** in the list linked to [4], we have only the surface n.14, and by changing (X_1, Y_1, Z_1) with $(-X_1, -Y_1, Z_1)$ is $\mathcal{H}_4 : Z_1^2 T_1 - X_1 Y_1 (X_1 + Y_1) = 0;$ this corresponds to the quartic surface

$$\mathcal{F}(4) : [ZT + (aX + bY)^2]^2 - TXY(X + Y) = 0, \quad 0 \neq a \neq b \neq 0.$$

For **case 2**, there are two surfaces in the list

- n.45, assuming $t = -1$ and exchanging (X, Y, Z) with $(X_1, Z_1, -Y_1)$, we have $\mathcal{H}_5 : Z_1^2 T_1 - X_1 Y_1 (X_1 + T_1) = 0,$ and

- n.64, exchanging (X_1, Y_1, Z_1) with $(-X_1, Z_1, -Y_1)$, we have $\mathcal{H}_6 : Z_1^2 T_1 - X_1 (X_1 Y_1 + T_1^2) = 0.$ The corresponding surfaces are

$$\mathcal{F}(5) : [ZT + (aX + bY)^2]^2 - TXY(X + T) = 0, \quad a \neq 0, b \neq 0,$$

$$\mathcal{F}(6) : [ZT + (aX + bY)^2]^2 - TX(XY + T^2) = 0, \quad a \neq 0, b \neq 0.$$

Finally, for **case 3**, we have the surfaces $\mathcal{H}$ in $\mathbf{P}_1^3$ that are almost-factorial because $\mathcal{H} \cdot \{T_1 = 0\}$ is irreducible. We thus find the surfaces n.72, n.76, and n.78. For the surface n.72, exchanging the coordinates (X_1, Y_1, Z_1) with $(-X_1, Z_1, Y_1)$, we obtain $\mathcal{H}_7 : Z_1^2 T_1 - X_1 (T_1 Y_1 + X_1^2) = 0,$ corresponding to the quartic

$$\mathcal{F}(7) : [ZT + (aX + bY)^2]^2 - TX(YT + X^2) = 0, \quad b \neq 0.$$

For n.78, exchanging (X_1, Y_1, Z_1) with $(-X_1, Z_1, -Y_1)$, we obtain

$\mathcal{H}_8 : Z_1^2 T_1 - (T_1^2 Y_1 + X_1^3) = 0,$ corresponding to

$$\mathcal{F}(8) : [ZT + (aX + bY)^2]^2 - T(YT^2 + X^3) = 0, \quad b \neq 0.$$

For n.76, exchanging (X_1, Y_1, Z_1) with $(Z_1 - Y_1, -X_1, Z_1 + Y_1)$, we have

$\mathcal{H}_9 : Z_1^2 T_1 - (T_1 Y_1^2 + X_1^3) = 0,$ which gives to the quartic

$$\mathcal{F}(9) : [ZT + (aX + bY)^2]^2 - T(Y^2T + X^3) = 0, \quad b \neq 0.$$

Now we need to examine each surface $\mathcal{F}(i)$, $4 \leq i \leq 9$, to obtain its suitable parametrization on $\mathbf{P}^2$ and the factor of non-regularity *M*. On each surface $\mathcal{F}_i$, $4 \leq i \leq 8$ every factor of M defines a plane that intersects $\mathcal{F}_i$ according to either a conic $\{ZT + (aX + bY)^2 = 0\}$ counted twice or the line $\{aX + bY = 0\}$ with multiplicity 4. According to Proposition 1, all such surfaces are 4-almost-factorial.

For the last surface $\mathcal{F}(9)$, the factor of non-regularity is T^5X^6. We have

$$\mathcal{F}(9) \cdot \{T = 0\} = 4\mathbf{r}, \quad \mathcal{F}(9) \cdot \{X = 0\} = \mathcal{C}_1 + \mathcal{C}_2,$$

where $\mathcal{C}_1 : \{X = 0 = ZT + b^2Y^2 - YT\}$, $\mathcal{C}_2 : \{X = 0 = ZT + b^2Y^2 + YT\}$.

Now it is

$$\mathcal{F}(9) \cap \{ZT + (aX + bY)^2 - YT = 0\} = \begin{cases} ZT + (aX + bY)^2 = YT \\ \qquad TX^3 \qquad\quad = 0 \end{cases}$$

$$= \{T = 0 = (aX + bY)^2\} + \{X^3 = 0 = \{ZT + (aX + bY)^2 - YT = 0\} = 2\mathbf{r} + 3\mathcal{C}_1.$$

On $\mathcal{F}(9)$, we consider the divisor of

$$\frac{(zt + (ax + by)^2 - yt)^2}{t} = \frac{(zt + (ax + by)^2)^2 - 2ty(zt + (ax + by)^2) + t^2y^2}{t}$$

$$= x^3 - 2y(zt + (ax + by)^2) + 2ty^2.$$

The cubic surface $\mathcal{S} : X^3 - 2Y(ZT + (aX + bY)^2) + 2TY^2 = 0$ gives

$$\mathcal{F}(9) \cdot \mathcal{S} = 4\mathbf{r} + 6\mathcal{C}_1 - 4\mathbf{r} = 6\mathcal{C}_1.$$

Then we consider $\mathcal{F}(9) \cap \{ZT + (aX + bY)^2 + YT = 0\}$. With a similar calculation, we find

$$\mathcal{F}(9) \cdot \{X^3 + 2Y(ZT + (aX + bY)^2) + 2TY^2 = 0\} = 6\mathcal{C}_2.$$

From Proposition 1, the surface $\mathcal{F}(9)$ is almost-factorial, and precisely is $\nu = 12$.

All these results are summarized in the following tables; ν = *index* of almost-factoriality, M = factor of non-regularity for the parametrization of the surface.

Almost-factorial quartic with $\mathcal{F}_\infty = 4\mathbf{r}$ and T dividing Δ

Equation of the surface	Parametrization τ	τ^{-1}	μ
$\mathcal{F}(4)$: $[TZ + (aX + bY)^2]^2$ $- TXY(X + Y) = 0$ $0 \neq a \neq b \neq 0$	$T : W^2U^2(W + U)^2$ $X : WU^2V^2(W + U)$ $Y : W^2UV^2(W + U)$ $Z : WUV^2(W + U) - V^4[aU + bW]^2$	$W : TY$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^5X^2Y^2(X + Y)^2$	4
$\mathcal{F}(5)$: $[TZ + (aX + bY)^2]^2$ $- TXY(X + T) = 0$ $a \neq 0,\ b \neq 0$	$T : U^2W^2(W - U)^2$ $X : W^2U^3(W - U)$ $Y : WUV^2(W - U)^2$ $Z : W^2U^2V(W - U)$ $- [aWU^2 + bV^2(W - U)]^2$	$W : T(T + X)$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^7X^2(X + T)^2$	4
$\mathcal{F}(6)$: $[TZ + (aX + bY)^2]^2$ $- TX(XY + T^2) = 0$ $a \neq 0,\ b \neq 0$	$T : W^2U^4$ $X : WU^5$ $Y : W^2U^2(V^2 - WU)$ $Z : WVU^4 - [aU^3 + bW(V^2 - WU)]^2$	$W : T^2$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^7X^4$	4

$\mathcal{F}(7)$: $[TZ + (aX + bY)^2]^2$ $- TX(TY + X^2) = 0$ $b \neq 0$	$T : W^6$ $X : W^4(U^2 - WV)$ $Y : W^3V(U^2 - WV)$ $Z : W^3U(U^2 - WV) -$ $-(U^2 - WV)^2)(aW + bV)^2$	$W : TX$ $U : TZ + (aX + bY)^2$ $V : TY$ $M = T^5X^6$	4
$\mathcal{F}(8)$: $[TZ + (aX + bY)^2]^2$ $- T(T^2Y + X^3) = 0$ $b \neq 0$	$T : W^6$ $X : W^5U$ $Y : W^3(WV^2 - U^3)$ $Z : W^5V - [aW^2U + b(WV^2 - U^3)]^2$	$W : T^2$ $U : TX$ $V : ZT + (aX + bY)^2$ $M = T^{11}$	4
$\mathcal{F}(9)$: $[TZ + (aX + bY)^2]^2$ $- T(TY^2 + X^3) = 0$ $b \neq 0$	$T : W^6$ $X : W^4(U^2 - V^2)$ $Y : W^3V(U^2 - V^2)$ $Z : W^3U(U^2 - V^2) -$ $-(U^2 - V^2)^2(aW + bV)^2$	$W : TX$ $U : ZT + (aX + bY)^2$ $V : TY$ $M = T^5X^6$	12

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